Gaussian approximation of moments of sums of independent symmetric random variables with logarithmically concave tails*

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Abstract

We study how well moments of sums of independent symmetric random variables with logarithmically concave tails may be approximated by moments of Gaussian random variables.

Let $\varepsilon_1, \varepsilon_2, \ldots$ be a Bernoulli sequence, i.e. a sequence of independent symmetric variables taking values ± 1. Hitczenko [4] showed that for $p \geq 2$ and $S = \sum_i a_i \varepsilon_i$,

$$\|S\|_p \sim \sum_{i \leq p} a_i^* + \sqrt{p}\Big(\sum_{i>p} (a_i^*)^2\Big)^{1/2} \tag{1}$$

where (a_i^*) denotes the nonincreasing rearrangement of $(|a_i|)$ and $f(p) \sim g(p)$ means that there exists a universal constant C such that $C^{-1}f(p) \leq g(p) \leq Cf(p)$ for any parameter p (see also [8] and [5] for related results). Gluskin and Kwapień [2] generalized the result of Hitczenko and found two sided bounds for moments of sums of independent symmetric random variables with logarithmically concave tails (we say that X has logarithmically concave tails if $\ln \mathbf{P}(|X| \geq t)$ is concave from $[0, \infty)$ to $[-\infty, 0]$). In particular they showed that for a sequence $(\mathcal{E}_i)$ of independent symmetric exponential random variables with variance 1 (i.e. the density $2^{-1/2}\exp(-\sqrt{2}|x|)$), $S = \sum_i a_i \mathcal{E}_i$, and $p \geq 2$,

$$\|S\|_p \sim p\|a\|_\infty + \sqrt{p}\|a\|_2, \tag{2}$$

where $\|a\|_p = (\sum_i |a_i|^p)^{1/p}$ for $1 \leq p < \infty$ and $\|a\|_\infty = \sup |a_i|$. Two sided inequality for moments of sums of arbitrary independent symmetric random variables was derived in [7].

Results (1) and (2) suggest that if all coefficients are of order $o(1/p)$ then $\|S\|_p$ should be close to the p-th norm of the corresponding Gaussian sum that is to $\gamma_p\|a\|_2$, where $\gamma_p = \|\mathcal{N}(0,1)\|_p = 2^{p/2}\Gamma(\frac{p+1}{2})/\sqrt{\pi}$. The purpose of our note is to verify this assertion.

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First we show the intuitive result that in the class of normalized symmetric random variables with logarithmically concave tails Bernoulli and exponential random variables are extremal.

Proposition 1. *Let X_i be independent symmetric r.v.'s with logarithmically concave tails such that $\mathbf{E}X_i^2 = 1$. Then for any $p \geq 3$,*

$$\Big\| \sum_{i=1}^n a_i \varepsilon_i \Big\|_p \leq \Big\| \sum_{i=1}^n a_i X_i \Big\|_p \leq \Big\| \sum_{i=1}^n a_i \mathcal{E}_i \Big\|_p .$$

Proof. Lower bound follows from Theorem 1.1 of [1] (in fact we do not use here the assumption of logconcavity of tails). To prove the upper bound it is enough to show that for all $a, b \in \mathbb{R}$ and $p \geq 3$,

$$\mathbf{E}|a + bX_i|^p \leq \mathbf{E}|a + b\mathcal{E}_i|^p .$$

Let $\varphi(x) = \frac{1}{2}(|a+bx|^p + |a-bx|^p)$, then φ' is convex on $[0,\infty)$ with $\varphi'(0) = 0$. Since $\mathbf{E}X_i^2 = 1 = \mathbf{E}\mathcal{E}_i^2$ there exist t_0 such that $\mathbf{P}(|X_i| \geq t_0) = \mathbf{P}(|\mathcal{E}_i| \geq t_0)$. Logconcavity of tails implies that $\mathbf{P}(|X_i| \geq t) \leq \mathbf{P}(|\mathcal{E}_i| \geq t)$ for $t \geq t_0$ and the opposite inequality holds for $0 \leq t \leq t_0$. Let $\varphi'(t_0) = ct_0$ for some $c > 0$. Then by convexity of φ' we have $(\varphi'(t) - ct)(\mathbf{P}(|\mathcal{E}_i| \geq t) - \mathbf{P}(|X_i| \geq t)) \geq 0$ for all t. Thus

$$\begin{aligned} 0 &\leq \int_0^\infty (\varphi'(t) - ct)(\mathbf{P}(|\mathcal{E}_i| \geq t) - \mathbf{P}(|X_i| \geq t))dt \\ &= \mathbf{E}(\varphi(\mathcal{E}_i) - \varphi(X_i)) - \frac{c}{2}\mathbf{E}(\mathcal{E}_i^2 - X_i^2) = \mathbf{E}|a + b\mathcal{E}_i|^p - \mathbf{E}|a + bX_i|^p . \end{aligned}$$

$\square$

Next technical lemma will be used to compare characteristic functions of Bernoulli and exponential sums.

Lemma 1. *Let $|a_1| \geq |a_2| \geq \ldots \geq |a_n|$. Then for any t,*

$$\prod_{i=1}^n \cos(a_i t) + \frac{1}{2}a_1^2 t^2 \geq \prod_{i=2}^n \frac{1}{1 + a_i^2 t^2/2} . \tag{3}$$

Proof. We will consider 3 cases.

Case I $|a_1 t| \leq \sqrt{2}$. Let $x_i = a_i^2 t^2/2$, then since $\cos(a_i t) \geq 1 - a_i^2 t^2/2 \geq 0$, to establish (3) it is enough to show that

$$\prod_{i=1}^n (1 - x_i) + x_1 \geq \prod_{i=2}^n \frac{1}{1 + x_i} \text{ for } 1 \geq x_1 \geq x_2 \geq \ldots \geq x_n \geq 0.$$

However,

$$\begin{aligned} \prod_{i=2}^n (1 + x_i)\Big[\prod_{i=1}^n (1 - x_i) + x_1\Big] &= (1 - x_1)\prod_{i=2}^n (1 - x_i^2) + x_1 \prod_{i=2}^n (1 + x_i) \\ &\geq (1 - x_1)(1 - \sum_{i=2}^n x_i^2) + x_1(1 + \sum_{i=2}^n x_i) \geq 1 - \sum_{i=2}^n x_i^2 + \sum_{i=2}^n x_1 x_i \geq 1. \end{aligned}$$

Case II $\sqrt{2} \le |a_1 t| \le \pi/2$. Then

$$\prod_{i=1}^n \cos(a_i t) + \frac{1}{2}a_1^2 t^2 \ge \frac{1}{2}a_i^2 t^2 \ge 1 \ge \prod_{i=2}^n \frac{1}{1+a_i^2t^2/2}.$$

Case III $|a_1 t| \ge \pi/2$. Then

$$\prod_{i=1}^n \cos(a_i t) + \frac{1}{2}a_1^2 t^2 \ge \frac{1}{2}a_i^2 t^2 - |\cos(a_1 t)| \ge 1 \ge \prod_{i=2}^n \frac{1}{1+a_i^2t^2/2}.$$

$\square$

Using the above lemma we may now compare moments of Bernoulli and exponential sums in the special case $p \in [2,4]$.

Lemma 2. *Let $|a_1| \ge |a_2| \ge \ldots \ge |a_n|$. Then for any $2 \le p \le 4$,*

$$\mathbf{E}\Big|\sum_{i=1}^n a_i\varepsilon_i\Big|^p \ge \mathbf{E}\Big|\sum_{i=2}^n a_i\mathcal{E}_i\Big|^p. \tag{4}$$

Proof. Let $S_1 = \sum_{i=1}^n a_i\varepsilon_i$ and $S_2 = \sum_{i=2}^n a_i\mathcal{E}_i$, obviously we may assume that $2 < p < 4$. By Lemma 4.2 of [3] we have for any random variable X with finite fourth moment,

$$\mathbf{E}|X|^p = C_p \int_0^\infty \Big(\varphi_X(t) - 1 + \frac{1}{2}t^2\mathbf{E}|X|^2\Big)t^{-p-1}dt.$$

where φ_X is the characteristic function of X and $C_p = -\frac{2}{\pi}\sin(\frac{p\pi}{2})\Gamma(p+1) > 0$. Notice that by Lemma 1,

$$\varphi_{S_1}(t) - \varphi_{S_2}(t) = \prod_{i=1}^n \cos(a_i t) - \prod_{i=2}^n \frac{1}{1+a_i^2t^2/2} \ge -a_1^2t^2/2,$$

thus

$$\mathbf{E}|S_1|^p - \mathbf{E}|S_2|^p = C_p \int_0^\infty \Big(\varphi_{S_1}(t) - \varphi_{S_2}(t) + a_1^2t^2/2\Big)t^{-p-1}dt \ge 0.$$

$\square$

To generalize the above result to arbitrary $p > 2$ we need one more easy estimate.

Lemma 3. *For any real numbers a, b we have*

$$\mathbf{E}|a\mathcal{E} + b|^p = |b|^p + \frac{p(p-1)}{2}a^2\mathbf{E}|a\mathcal{E}+b|^{p-2} \text{ for } p \ge 2 \tag{5}$$

and

$$\mathbf{E}|a\varepsilon + b|^p \ge |b|^p + \frac{p(p-1)}{2}a^2|b|^{p-2} \text{ for } p \ge 3. \tag{6}$$

Proof. By integration by parts it is easy to show that for any $f \in C^2(\mathbb{R})$ of at most polynomial growth we have $\mathbf{E}f(\mathcal{E}) = f(0) + \frac{1}{2}\mathbf{E}f''(\mathcal{E})$. If we take $f(x) = |ax + b|^p$ we obtain (5). To prove (6) it is enough to notice that the function $g(x) := \mathbf{E}|x\varepsilon + b|^p$ satisfies $g(0) = |b|^p$, $g'(0) = 0$ and $g''(x) = p(p-1)\mathbf{E}|x\varepsilon + b|^{p-2} \geq p(p-1)|b|^{p-2}$. □

Our first theorem shows that moments of Bernoulli sums dominate moments of exponential sums up to few largest coefficients.

Theorem 1. *Let $|a_1| \geq |a_2| \geq \ldots \geq |a_n|$. Then for any $p \geq 2$,*

$$\gamma_p^p\Big(\sum_{i=1}^n a_i^2\Big)^{p/2} \geq \mathbf{E}\Big|\sum_{i=1}^n a_i\varepsilon_i\Big|^p \geq \mathbf{E}\Big|\sum_{i=\lceil p/2\rceil}^n a_i\mathcal{E}_i\Big|^p \geq \gamma_p^p\Big(\sum_{i=\lceil p/2\rceil}^n a_i^2\Big)^{p/2}. \tag{7}$$

Proof. To establish the middle inequality we will show by double induction first on k then on n that for $p \in (2k, 2k+2]$,

$$\mathbf{E}\Big|\sum_{i=1}^n a_i\varepsilon_i\Big|^p \geq \mathbf{E}\Big|\sum_{i=k+1}^n a_i\mathcal{E}_i\Big|^p. \tag{8}$$

For $k = 1$ this follows by Lemma 2. Suppose that our assertion holds for $k-1$ and let $p \in (2k, 2k+2]$. For $n < k+1$ the inequality (7) is obvious. If $n \geq k+1$ and (8) holds for $n-1$ then by (6), induction assumption, and (5),

$$\begin{aligned}
\mathbf{E}\Big|\sum_{i=1}^n a_i\varepsilon_i\Big|^p &\geq \mathbf{E}\Big|\sum_{i=2}^n a_i\varepsilon_i\Big|^p + a_1^2\frac{p(p-1)}{2}\mathbf{E}\Big|\sum_{i=2}^n a_i\varepsilon_i\Big|^{p-2} \\
&\geq \mathbf{E}\Big|\sum_{i=k+2}^n a_i\mathcal{E}_i\Big|^p + a_{k+1}^2\frac{p(p-1)}{2}\mathbf{E}\Big|\sum_{i=k+1}^n a_i\varepsilon_i\Big|^{p-2} \\
&= \mathbf{E}\Big|\sum_{i=k+1}^n a_i\mathcal{E}_i\Big|^p.
\end{aligned}$$

First inequality in (7) follows by the Khintchine inequality with optimal constant [3] and the last inequality in (7) is an easy consequence of the fact that $\mathcal{E}$ is a mixture of gaussian r.v.'s (see Remark 5 in [6]). □

Next two corollaries present more precise versions of inequalities (1) and (2).

Corollary 1. *For any $p \geq 2$ we have*

$$\begin{aligned}
\max\Big\{\gamma_p\Big(\sum_{i\geq\lceil p/2\rceil}(a_i^*)^2\Big)^{1/2}, \frac{1}{\sqrt{2}}\sum_{i<\lceil p/2\rceil}a_i^*\Big\} &\leq \Big\|\sum_{i=1}^n a_i\varepsilon_i\Big\|_p \\
&\leq \gamma_p\Big(\sum_{i\geq\lceil p/2\rceil}(a_i^*)^2\Big)^{1/2} + \sum_{i<\lceil p/2\rceil}a_i^*.
\end{aligned}$$

Proof. We have by the triangle inequality and the Khintchine inequality with optimal constant [3],

$$\begin{aligned}\Big\|\sum_{i=1}^n a_i\varepsilon_i\Big\|_p &\le \Big\|\sum_{i\ge\lceil p/2\rceil} a_i^*\varepsilon_i\Big\|_p + \Big\|\sum_{i<\lceil p/2\rceil} a_i^*\varepsilon_i\Big\|_p \\ &\le \gamma_p\Big(\sum_{i\ge\lceil p/2\rceil}(a_i^*)^2\Big)^{1/2} + \sum_{i<\lceil p/2\rceil} a_i^*.\end{aligned}$$

To show the lower bound we use (7)

$$\Big\|\sum_{i=1}^n a_i\varepsilon_i\Big\|_p = \Big\|\sum_{i=1}^n a_i^*\varepsilon_i\Big\|_p \ge \gamma_p\Big(\sum_{i\ge\lceil p/2\rceil}(a_i^*)^2\Big)^{1/2}$$

and an easy estimate

$$\Big\|\sum_{i=1}^n a_i\varepsilon_i\Big\|_p \ge \Big\|\sum_{i<\lceil p/2\rceil} a_i^*\varepsilon_i\Big\|_p \ge \big(\mathbf{P}(\varepsilon_i = 1 \text{ for } 1\le i<\lceil p/2\rceil)\big)^{1/p}\sum_{i<\lceil p/2\rceil} a_i^*.$$

□

Corollary 2. *For any $p \ge 2$ we have*

$$\max\Big\{\gamma_p\|a\|_2, \frac{p}{e\sqrt{2}}\|a\|_\infty\Big\} \le \Big\|\sum_{i=1}^n a_i\mathcal{E}_i\Big\|_p \le \gamma_p\|a\|_2 + p\|a\|_\infty.$$

Proof. Let $S = \sum_{i=1}^n a_i\mathcal{E}_i$ and $k = \lceil p/2\rceil - 1$. We have $\|S\|_p \ge \gamma_p\|a\|_2$ by the last inequality in (7). Moreover

$$\|S\|_p \ge \|a\|_\infty\|\mathcal{E}\|_p = \|a\|_\infty\frac{1}{\sqrt{2}}(\Gamma(p+1))^{1/p} \ge \frac{p}{\sqrt{2}e}\|a\|_\infty.$$

To get the upper bound we use twice bounds (7) and obtain

$$\begin{aligned}\|S\|_p - \gamma_p\|a\|_2 &\le \|S\|_p - \Big\|\sum_{i>k} a_i^*\mathcal{E}_i\Big\|_p \le \Big\|\sum_{i\le k} a_i^*\mathcal{E}_i\Big\|_p \le \|a\|_\infty\Big\|\sum_{i\le k}\mathcal{E}_i\Big\|_p \\ &\le \|a\|_\infty\Big\|\sum_{i\le 2k}\varepsilon_i\Big\|_p \le 2k\|a\|_\infty \le p\|a\|_\infty.\end{aligned}$$

□

Now we may state a result that generalizes (up to a multiplicative constant) previous corollaries.

Theorem 2. *Let X_i be independent symmetric r.v.'s with logarithmically concave tails such that $\mathbf{E}X_i^2 = 1$ and $|a_1| \geq |a_2| \geq \ldots \geq |a_n|$. Then for any $p \geq 3$,*

$$\max\Big\{\gamma_p\Big(\sum_{i\geq\lceil p/2\rceil} a_i^2\Big)^{1/2}, \Big\|\sum_{i<p} a_iX_i\Big\|_p\Big\} \leq \Big\|\sum_{i=1}^n a_iX_i\Big\|_p$$

$$\leq \gamma_p\Big(\sum_{i\geq\lceil p/2\rceil} a_i^2\Big)^{1/2} + \Big\|\sum_{i<p} a_iX_i\Big\|_p.$$

Proof. Lower bound is an immediate consequence of Theorem 1 and Proposition 1. To get the upper bound let $k = \lceil p/2\rceil - 1$. Then

$$\Big\|\sum_{i=1}^n a_iX_i\Big\|_p \leq \Big\|\sum_{i>2k} a_iX_i\Big\|_p + \Big\|\sum_{i\leq 2k} a_iX_i\Big\|_p \quad \leq \gamma_p\Big(\sum_{i>k} a_i^2\Big)^{1/2} + \Big\|\sum_{i\leq 2k} a_iX_i\Big\|_p$$

again by Theorem 1 and Proposition 1. □

Remark. By the result of Gluskin and Kwapień we have

$$\Big\|\sum_{i<p} a_iX_i\Big\|_p \sim \sup\Big\{\sum_{i<p} a_ib_i \colon \sum_{i<p} M_i(b_i) \leq p\Big\},$$

where $M_i(x) = x^2$ for $|x| \leq 1$ and $M_i(x) = -\ln \mathbf{P}(|X_i| \geq x)$ for $|x| > 1$.

We conclude with one more result about Gaussian approximation of moments.

Corollary 3. *Let X_i be as in Theorem 2, then for any $p \geq 3$,*

$$\Big|\Big\|\sum_{i=1}^n a_iX_i\Big\|_p - \gamma_p\|a\|_2\Big| \leq p\|a\|_\infty.$$

Proof. The statement immediately follows by Proposition 1 and Corollaries 1 and 2. □

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